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A parametric dynamic tuning system for the diatonic scale

Towards five-limit just intonation based on a hexachord analysis algorithm

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Contents

Abstract

The subject of this document is mainly tuning. It focusses on tuning within a digital environment and adds some thoughts about developing cognition models to automate modulations, up to just intonation. The first part concerns a two-dimensional tuning system that first had been created in 2012, before evolving into a library with separated classes for various functions. It's a dynamic tuning system that is suited for the use of mean tone temperament, Pythagorean tuning, and more. The system is based on $Ts()$ values, a proposed two-dimensional numerical pitch representation system, which can be easily translated into MIDI note numbers and note names as well. In the second part a more sophisticated system will be discussed, based on three variable steps within a scale. The ideas about pattern matching and key-finding algorithms and status of this current research will be mentioned. The main areas this research on tuning covers are music cognition, mathematics, and music theory, especially medieval.

This document is written for the Musikinformatik department of the Johannes Gutenberg Universität Mainz. As might be clear by now it is written in English, although for convenience of the German reader the German note naming system will be used throughout this document, unless explicitely otherwise stated. *His* instead of *B-sharp* and *B* instead of *B-flat*, that is. The shorthand *e* for *e-moll* in figure 2 represents the key of *E minor*. Also, the German translation of some specific terms will be added as a footnote.

The theories to be described here are or will be implemented in a multidimensional tuning library, named *Muditulib* (Seelen, 2014). The term *multidimensional* needs a little explanation here, for it is used differently in closely related contexts. The dimensions meant here represent the number of different steps or variables in a scale, e.g. the semitone[1] and whole tone[2] in the diatonic scale, as is shown in table 1. This in contrast to multidimensionality within the scope of just intonation, where an n-dimensional space represents the nth prime number to which the frequency ratios are limited, starting with 3, 5, 7, 11 for the first, second, third and fourth dimension, respectively (see Keislar, 1987; Fokker, 1968). Nevertheless, prime numbers and just intonation will be discussed in the second part of this document. The research project is not finished yet, although it is getting close to completion by now.

Fbb	Gbb	Abb	Bb	Cb	Db
Ebb	Fb	Gb	Ab	B	C
Db	Eb	F	G	A	H
C	D	E	F#	G#	A#
H	C#	D#	E#	F##	G##

Table 1: Two-dimensional representation of the German note naming system. One horizontal step is one whole tone, one vertical step a semitone.

The goal of the project has been the development of a dynamic tuning system that both respects the diatonic scale, the basis of many western music compositions from the early middle ages until now, and, as much as eventually possible, the harmonic series. Of course this is not a new idea. Inventing the perfect tuning system has been a research subject for over two millenniums (see Barbour, 2004). A huge part of the problem has been the lack of flexibility of physical instruments, which vanished with the

[1] Halbtonschritt
[2] Ganztonschritt

introduction of computer music. But also for digital instruments it is hard to fit the needs of the diatonic scale as well as small integer relationships of frequencies exactly into a tuning system, especially when the music is polyphonic and when it changes tonality constantly, as is often the case. Still nowadays the standard in digital music production is the twelve tone equal temperament (12-TET), the compromise which could said to be obsolete. However, the reason for its remaining powerful existence nowadays is not the topic of this document.

The first part of this document describes a two-dimensional tuning system based on the two variables semitone and whole tone. An important feature of the system is the variable ratio between the size of both steps. Another important part is the mapping of MIDI note values, a one-dimensional scale, to these two dimensions. Apart from the tuning system itself, a numeric two-dimensional pitch representation system will be proposed and its relation to the MIDI note system will be explained. This mapping can be done quite easily, as shall be explained, when requesting certain key information from the user, although automation of this parameter requires algorithms that can get complex. The problem touches the subject *pitch spelling*, the pitch translation of piano roll data to traditional music notation. It has been dealt with by several researchers independently (see Longuet-Higgins and Steedman, 1971; Temperley, 2004; Cambouropoulos, 2003; Meredith, 2003; Chew and Chen, 2005; Honingh, 2006). To Gerard van Wolferen, senior lecturer at the Utrecht School of Music and Technology, however, none of the resulting algorithms were completely satisfying, because to him they were all based too much on mathematical instead of music theoretical and compositional models. At the end of 2012 he requested the present writer to develop an algorithm that was based on hexachord analysis, a pattern matching approach using Guido of Arezzo's hexachord pattern (see Grout and Palisca, 1988, ch. 2). Apart from the different approach of pitch spelling such a hexachord analysis supplies additional data that could be valuable for tuning purposes. In the second part the hexachord analysis as well as its use for tuning systems will be discussed, and a tuning system based on three variables proposed.

1 A parametric dynamic tuning system for the diatonic scale

The system described here is based on the diatonic scale, in contrast to the chromatic scale. A concise description of this music compostion tool will be given in the next paragraph.

1.1 The diatonic scale

The terms chromatic and diatonic descend from the old Greek musical system. Together with the enharmonic they formed the three tetrachords the Greek musical scales were made up from (Grout and Palisca, 1988, ch. 1). The ancient tuning theory is clearly described by J. Murray Barbour (see Barbour, 2004, ch. II). Today's use of those terms is somehow related to that of their namegivers, although the tetrachord itself lost its value. The diatonic scale is a scale that consists of seven intervals or steps. The eighth note, or the octave, is a repetition of the first one, usually with a frequency ratio of 2 : 1. Those seven steps are divided into five larger ones, the whole tones, and two smaller ones, the semitones. The scale then created is actually the same as two Greek diatonic tetrachords on top of each other, at one side overlapping (*conjunct*) and at the other side separated by one whole tone (*disjunct*). By the chromatic scale, however, usually a division of the octave in twelve equal parts is meant, which is quite different from an accumulation of Greek chromatic tetrachords.

So far I described the historical outlines of the system. In the frequency domain the relation between the octave x and whole tone T and semitone s is as shown in equation 1, where $1 < s < T$. The relation between the octave and the chromatic step c is shown in equation 2.

$$x = T^5 \cdot s^2 \tag{1}$$

$$x = c^{12} \tag{2}$$

The diatonic scale is often characterized by a sequence of seven adjacent[3] alphabetical characters to represent a certain pitch, placed between every two steps in the scale. In table 2 the scales on C and F are shown compared to a relative note naming system consisting of short syllables[4]. In the *do-re-mi* pattern the two semitones are located between *mi* and *fa* and between *ti* and *do.* If this relative pattern is shifted along the absolute scale the root characters are sharpened or flattened for every shift on the line of fifths[5], as is inherent to the construction of the pattern. That is, if G is set to the new *do*, then F is sharpened to *Fis*, the new *ti.* Within the diatonic scale, apart from their eventual alterations as just described, all seven root characters occur only once. Each step in the scale means an alteration of the root character to an adjacent one. The German system however consists of eight characters, where H and B represent the same root, so an alteration from H to B is not a step, but just a flattening. The exception is shown in

[3] *A* comes after *G* and the *H* is a replacement of the hard *B* in the German system.

[4] Here the *do-re-mi* scale is used relatively to any pitch or alphabetical character, although in some countries, e.g. Belgium and France, it is used in absolute form instead of *C-D-E*.

[5] *..-B-F-C-G-D-A-E-H-..*

C	D	E	F	G	A	H	C
do	re	mi	fa	sol	la	ti	do
F	G	A	B	C	D	E	F

Table 2: The diatonic note naming system

					Ut	Re	Mi	Fa	Sol	La
hard					G	A	H	C	D	E
	Ut	Re	Mi	Fa	Sol	La				
natural	C	D	E	F	G	A				
				Ut	Re	Mi	Fa	Sol	La	
soft				F	G	A	B	C	D	

Table 3: Relation between natural, hard, and soft hexachord. The soft B takes the same position - stepwise - as the hard H and therefore has the same root.

equations 3 and 4. Thus, the interval *H:B* is not a semitone, but in fact a whole tone minus a semitone. Their relation is shown in table 3, which shows the ancient hexachord relationships between the hard (*durum*), natural, and soft (*mollum*) hexachords.

$$B \neq H \cdot \frac{1}{s} \tag{3}$$

$$B = H \cdot \frac{s}{T} \tag{4}$$

All intervals within the diatonic scale are then made up from a number of whole tones and semitones. Table 4 shows a large selection of possible intervals. The intervals that don't fit into one *do-re-mi* scale[6] are emphasized.

1.2 The Ts classes

Now the diatonic scale and its two variables or dimensions are explained it is time to propose a numeric notation method, in order to perform calculations on the data. The notation is a two-dimensional numeric reference to note names and their octave designation, without losing data. This in contrast to one-dimensional representations like the MIDI note system, which lacks information about the root note (alphabetical character) and the line-of-fifths representation proposed by David Temperley (see Temperley, 2004, ch. 5), which lacks information about the octave designation. A $Ts()$ value consists

[6]That is, they are formed by two non-adjacent hexachords.

interval	T	s
prime	0	0
minor second	0	1
major second	1	0
minor third	1	1
major third	2	0
diminished fourth	1	2
perfect fourth	2	1
augmented fourth	3	0
diminished fifth	2	2
perfect fifth	3	1
minor sixth	3	2
major sixth	4	1
diminished seventh	3	3
minor seventh	4	2
major seventh	5	1
octave	5	2

Table 4: A selection of intervals within the diatonic scale and the number of whole tones T and semitones s they are made up from.

of a number of whole tones and a number of semitones, respectively. The reference chosen for the proposed $Ts()$ value, however, is the MIDI note system, for translations between both should be as easy as possible. Therefore, $Ts(0,0)$ is set equal to the note C meant by MIDI note 0. Middle C, five octaves higher, is then represented by $Ts(25,10)$, for each octave higher equals an addition of $(5,2)$. The enharmonically equal *His* is represented by $Ts(26,8)$, one step less away from the reference C[7], as the root character H shows. The value $Ts(T_n, s_n)$ can be easily translated to a MIDI note number, as is shown in equation 5, where m is the MIDI note number and T_n and s_n represent the numbers of whole tones and semitones, respectively. Both *His* and *C'* refer to MIDI note 60.

$$m = 2 \cdot T_n + s_n \tag{5}$$

The Ts tuning system is divided into three classes for the moment: [midi2ts], [ts2symbol], and [ts2freq].

1.2.1 [ts2symbol]

As stated before a $Ts()$ value respresents diatonic pitch data, i.e. a note name, without losing any of the data. This means that a back and forth

[7]Total steps equals the sum of whole tones and semitones: $26 + 8 = 34$, compared to $25 + 10 = 35$.

translation from $Ts()$ to note names with their octave designation should be possible. The sum of whole tones and semitones, the total amount of steps from the reference note C, determines the root note. Flattening or sharpening can then be calculated from the amount of semitones compared to whole tones, after subtracting $(5, 2)$ for each octave higher than the reference. Figure 1 shows the result of the translation of both values $Ts(29, 10)$ and $Ts(28, 12)$ to a note name symbol. The Lilypond[8] method of expressing note names is used, for the output should be easily used as input for Lilypond. One important difference is that Lilypond doesn't accept more than two accidentals for one note name, while [ts2symbol] does.

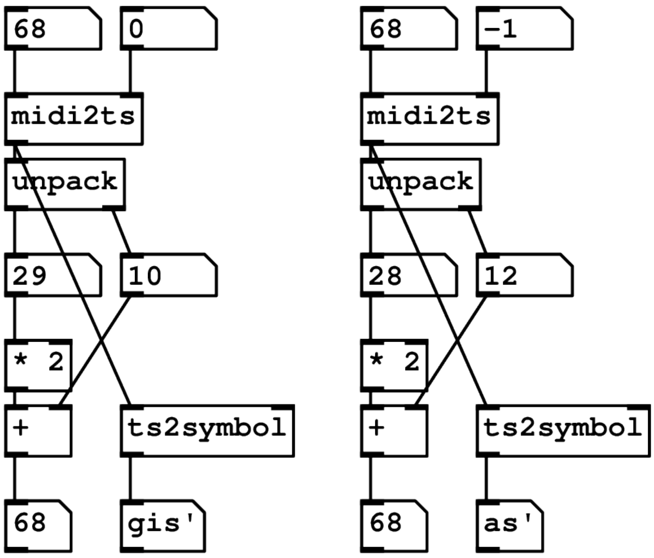

Figure 1: Example of [ts2symbol] in Pure Data.

1.2.2 [ts2freq]

Since the initial goal was to develop a tuning system this class is an important one. The way to make a tuning system respect the diatonic scale at most is to separate the whole tones from the semitones, just as the music notation prescribes. This has been common practice through the ages, as the tuning examples in the next section will show. The tuning system described here uses a variable semitone to whole tone ratio r. The relation between r and the frequency ratios T and s, for whole tones and semitones is shown in equation 6. The translation of the value $Ts(T_n, s_n)$ to a frequency is represented by equation 7, where f means the frequency in cycles per

[8]LilyPond is a music engraving program, free software at `http://www.lilypond.org/`.

Piano keys	Es	B		Fis	Cis	Gis
major/dur	B	F	C	G	D	A
minor/moll	g	d	a	g	d	a

Table 5: Piano keys and the major and minor keys they enable.

second, A is the reference frequency of $Ts(29, 11)$[9], and x is the frequency ratio of an octave.

$$r = \frac{\log s}{\log T} \tag{6}$$

$$f = x^{((T_n - 29) + (s_n - 11) \cdot r)/(5 + 2 \cdot r)} \cdot A \tag{7}$$

1.2.3 [midi2ts]

The translation of the one-dimensional MIDI note data to the two-dimensional $Ts()$ can be done in several ways. One possibility is for example a fixed mapping procedure. Every MIDI note is then fixed to a $Ts()$ value and can in that way not be altered. The [midi2ts] class fixes this mapping partly. That is, the mapping is defined by the user and can be altered manually. Translated into the musical domain this means that the user defines the black keys of a piano keyboard for example to be *Cis*, *Es*, *Fis*, *Gis*, and *B*. However, this can be altered at any moment. Within one modulation setting, controlled by parameter *mod*, six adjacent major and three minor keys can be modulated to. For the given example setting, referred to by *mod* if set to 0, these keys are shown in table 5. The minor keys request more due to their leading tone which lies outside the natural key[10]. By increasing *mod* one of twelve keys in the octave is altered for every integer, starting with *Es* to *Dis* for MIDI note 3, up the circle of fifths, i.e. *B* to *Ais* for MIDI note 10. The alterations happen by adding $(1, -2)$ to the $Ts()$ mapped to the note to be altered. Of course, by decreasing *mod*, everything happens in the opposite direction, starting with *Gis* to *As* for MIDI note 8, and then down the circle of fifths. Clearly, a dynamic form of mapping could be done if one could roughly derive the musical key on a certain spot from the sequence of MIDI notes itself. Such algorithms will be discussed in the second part.

1.3 Tunings and temperaments

The tuning technique of making a clear distinction between the whole tone and the semitone is as old as the diatonic scale. Examples of such two-

[9]This is the A above middle C, just like - in most cases - MIDI note number 69 is.

[10]E.g. *a-moll* requires *Gis*.

dimensional tunings are Pythagorean tuning and meantone temperament. The difference between both can be summarized to a difference in ratio r.

1.3.1 Pythagorean

Pythagorean tuning is based on an octave ratio of 2 : 1 and a perfect fifth ratio of 3 : 2. It is called a tuning for its use of merely integer relationships in all ratios. The tuning is called *three-limit* for all frequency ratios are limted to prime number three. The whole tone T is the result of two fifths minus one octave, thus has a frequency ratio of $2^{-3} \cdot 3^2 = \frac{9}{8}$. Therefore s has a frequency ratio $2^8 \cdot 3^{-5} = \frac{256}{243}$. Ratio r can then be calculated in two ways, of which the easiest one is shown in equation 8. This can however be expressed differently, that is, by giving the relation between both reference intervals, as is shown in equation 9. From this equation r can be derived as well, as is shown in equation 10.

$$r = \frac{\log s}{\log T} \approx 0.442474596 \tag{8}$$

$$\frac{3}{2} = 2^{(3+r)/(5+2 \cdot r)} \tag{9}$$

$$r = \frac{5 \cdot \log_2 \frac{3}{2} - 3}{1 - 2 \cdot \log_2 \frac{3}{2}} \approx 0.442474596 \tag{10}$$

1.3.2 Mean tone

Mean tone temperament is based on an octave ratio of 2 : 1 and a major third ratio of 5 : 4. It is called a temperament for its use of non-integer relationships in some ratios. The major third is split into two equal whole tones, as shown in equation 11. This is called the mean tone. The relation between both reference intervals is shown in equation 12. From this equation r can be derived, as is shown in equation 13. The integer relationships are based only on the prime numbers two and five and *not* on the number three.

$$T = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} \tag{11}$$

$$\frac{5}{4} = 2^{2/(5+2 \cdot r)} \tag{12}$$

$$r = \frac{1}{\log_2 \frac{5}{4}} - 2.5 \approx 0.60628372 \tag{13}$$

1.3.3 Slightly stretched octave, perfect fifth and thirds

As can be concluded from the previous two examples within this tuning system two reference intervals can be chosen and the others automatically result from them. So, a third option would be to abandon the octave ratio in favor of both perfect fifth and major third. An additional advantage of this option is that the minor third would also be perfect, that is, a frequency ratio of 6 : 5. How r is derived is shown in equations 14 and 15 sequentially. How then x is derived is shown in equation 16. The integer relationships are based on the prime numbers two, three, and five. All intervals made up from thirds are then justly intonated[11]. T equals that of mean tone temperament, while s is a little bit larger in this temperament.

$$\frac{5}{4} = \left(\frac{3}{2}\right)^{2/(3+r)} \tag{14}$$

$$r = \frac{2}{\log_{\frac{3}{2}} \frac{5}{4}} - 3 \approx 0.634118985 \tag{15}$$

$$x = \left(\frac{5}{4}\right)^{(5+2\cdot r)/2} \approx 2.01246118 \tag{16}$$

1.3.4 n-TET equal temperaments

The idea for parameter r arose from the construction of several tuning systems that divide the octave in a number (n) equal parts. This total number of parts is a result of the combination of five times the number of parts to be used for a whole tone and twice the number to be used for a semitone. For example 31-TET is a cumulation of 3 and 5 parts for all semitones and whole tones, respectively. The semitone to whole tone ratio r then is $\frac{3}{5} = 0.6$. Table 6 shows a selection of such n-TET tuning systems. Figure 2 shows how the diatonic note names are related to the 31 parts.

2 Towards five-limit just intonation based on a hexachord analysis algorithm

In the first part of this document a system has been described into which the diatonic scale fits perfectly. However, the second part of the initial goal, that is, to respect the harmonic series as much as eventually possible, leaves much to be desired. Compared to ancient fixed tunings and temperaments, apart from easily switching between them by resetting parameter r, the $Ts()$

[11]E.g. primes, thirds, fifths, sevenths, ninths, elevenths, thirteenths, et cetera, but *not* their octave inversions of course.

```
0         1         2         3   mod:  usable keys:
01234567890123456789012345678901
||||||||||||||||||||||||||||||||
  |  | |  | |  |  | |  | |  | |         D, A, E,
  C# | D# | |  F# | G# | A# | |    4    H, Fis, Cis,
  |  D |  E E# |  G |  A |  H H#        h, fis, cis
  |  | |  | |  |  | |  | |  | |
--------------------------------
| |  | |  | |  |  | |  | |  |  |        G, D, A,
| C# | D# | |  F# | G# | A# |  |   3    E, H, Fis,
C |  D |  E E# |  G |  A |  H  C        e, h, fis
| |  | |  | |  |  | |  | |  |  |
--------------------------------
| |  | |  |  | |  | |  | |  |  |        C, G, D,
| C# | D# |  | F# | G# | A# |  |   2    A, E, H,
C |  D |  E  F |  G |  A |  H  C        a, e, h
| |  | |  |  | |  | |  | |  |  |
--------------------------------
| |  | |  |  | |  | |  |  | |  |        F, C, G,
| C# | D# |  | F# | G# |  B |  |   1    D, A, E,
C |  D |  E  F |  G |  A  | H  C        d, a, e
| |  | |  |  | |  | |  |  | |  |
--------------------------------
| |  |  | |  | |  | |  |  | |  |        B, F, C,
| C# |  Eb|  | F# | G# |  B |  |   0    G, D, A,
C |  D  | E  F |  G |  A  | H  C        g, d, a
| |  |  | |  | |  | |  |  | |  |
--------------------------------
| |  |  | |  | |  |  | |  | |  |        Es, B, F,
| C# |  Eb|  | F# |  Ab|  B |  |  -1    C, G, D,
C |  D  | E  F |  G  | A  | H  C        c, g, d
| |  |  | |  | |  |  | |  | |  |
--------------------------------
|  | |  | |  | |  |  | |  | |  |         As, Es, B,
|  Db|  Eb|  | F# |  Ab|  B |  |  -2     F, C, G,
C  | D  | E  F |  G  | A  | H  C         f, c, g
|  | |  | |  | |  |  | |  | |  |
--------------------------------
|  | |  | |  |  | |  | |  | |  |         Des, As, Es,
|  Db|  Eb|  |  Gb|  Ab|  B |  |  -3     B, F, C,
C  | D  | E  F  | G  | A  | H  C         b, f, c
|  | |  | |  |  | |  | |  | |  |
--------------------------------
|  | |  | |  |  | |  | |  |  | |         Ges, Des, As,
|  Db|  Eb|  |  Gb|  Ab|  B  | |  -4     Es, B, F,
C  | D  | E  F  | G  | A  |  CbC         es, b, f
|  | |  | |  |  | |  | |  |  | |
||||||||||||||||||||||||||||||||
01234567890123456789012345678901
0         1         2         3
```

Figure 2: Equal division of the octave in 31 parts on a logarithmic scale and the result of the *mod* parameter.

construction	n-TET	ratio r	relevance
7	7	1/1	
7+7+5	19	2/3	
7+7+7+5+5	31	3/5	(approx. mean tone)
7+7+7+7+5+5+5	43	4/7	
7+7+7+7+7+5+5+5+5	55	5/9	
...	...	...	
7+5	12	1/2	(chromatic scale)
...	...	...	
7+7+7+7+5+5+5+5+5	53	4/9	(approx. Pythagorean)
7+7+7+5+5+5+5	41	3/7	
7+7+5+5+5	29	2/5	
7+5+5	17	1/3	
5	5	0/1	

Table 6: A selection of n-TET systems and their construction, ordered to their r ratio.

based system's main advantage is that it can modulate outside boundaries, used to be circumvented by all sorts of temperaments[12], dynamically.

In this second part a refinement of the tuning system is proposed, however not completely elaborated. The final result of this part of the research project is expected in the second half of 2014. How and when it will be published is not certain yet. Its software implementation will also be a part *Muditulib*, and therefore hopefully ready at its initial release.

2.1 Just intonation, prime numbers, and *harmonicity*

"The Pythagorean tuning may be thought of as the limiting form of just intonation, since it has a great many pure fifths, but no pure major thirds" (Barbour, 2004, ch. V: Just intonation). Every integer, as well as every ratio consisting of two integers, can be represented as a product of prime numbers. Usually the name *just intonation* is used for tunings consisting of only integer relationships, and more precisely, tunings in which at least some of the ratios are made up from at least the prime number five. Generally small integer ratios are more harmonious than large integer ratios, however: "not only the smallness of the numbers of a relationship is relevant for the construction of a harmonic interval, but also their *divisibility*" (Barlow, 1987). Barlow explains that historically preferred numbers for intervals are based on these prime numbers two, three, and five. In contrast to the three-limit Pythagorean tuning, the application of this five-limit system results

[12]This was done by tempering some or more perfect intervals, to not make crossing modulation boundaries sound too harsh (see Barbour, 2004, ch. VII: Irregular systems).

in relatively simple ratios for all possible intervals in the diatonic system (see table 7), except for the augmented fourth and the diminished fifth, which are both generally considered rather dissonant. A. D. Fokker however, advocating the use of the seventh harmonic, joins Christiaan Huygens in his opinion that the intervals 7 : 5 and 10 : 7 should be reckoned among the consonants (Fokker and Pol, 1942, p. 475-476). Fokker states that the system rather than the seventh harmonic itself is to blame for its impurity. According to Barlow's proposed inversion of the sum of his *indigestibility* function, shown in equation 17, though, the intervals 7:5 and 45:32 are almost equally (and not particularly) harmonious: $\frac{1}{\xi(7)+\xi(5)} = 0.05993$ and $\frac{1}{\xi(45)+\xi(32)} = 0.05976$.

interval	T	t	s	frequency ratio	
				natural numbers	powers of primes
prime	0	0	0	1:1	
minor second	0	0	1	16:15	$2^4.5^{-1}.3^{-1}$
major second	0	1	0	10:9	$5.2.3^{-2}$
”	1	0	0	9:8	$3^2.2^{-3}$
minor third	1	0	1	6:5	$3.2.5^{-1}$
major third	1	1	0	5:4	5.2^{-2}
perfect fourth	1	1	1	4:3	$2^2.3^{-1}$
augmented fourth	2	1	0	45:32	$5.3^2.2^{-5}$
diminished fifth	1	1	2	64:45	$2^6.5^{-1}.3^{-2}$
perfect fifth	2	1	1	3:2	3.2^{-1}
minor sixth	2	1	2	8:5	$2^3.5^{-1}$
major sixth	2	2	1	5:3	5.3^{-1}
minor seventh	2	2	2	16:9	$2^4.3^{-2}$
”	3	1	2	9:5	$3^2.5^{-1}$
major seventh	3	2	1	15:8	$5.3.2^{-3}$
octave	3	2	2	2:1	2

Table 7: Intervals of the diatonic system in five-limit just intonation.

$$\xi(N) = 2\sum_{r=1}^{\infty}\left\{\frac{n_r(p_r-1)^2}{p_r}\right\} \tag{17}$$

where
$N = \prod_{r=1}^{\infty} p_r^n$
p is a prime, and
n is a natural number.

2.2 The Tts classes

As was already shown in table 7, the intervals are now divided into three variable parts. That is, the whole tones are divided into smaller (t) and larger (T) ones. This division is necessary to fulfill the needs of the five-limit just intonation. How the intervals are constructed from powers of prime numbers is shown in the rightmost column. As a rule of thumb, the ideal intervals consist of an equal amount of small and large whole tones, if the total amount of whole tones is even. If the total amount is odd, the amount of large whole tones exceeds the amount of small whole tones by one. The frequency ratios for all variable steps are $T = \frac{9}{8}$, $t = \frac{10}{9}$, and $s = \frac{16}{15}$. To define which whole tone in the diatonic scale should then be large and which should be small is a quite challenging task, because it depends on many factors. The difference between the large and the small whole tone is called the syntonic comma, which equals $\frac{T}{t} = \frac{81}{80}$.

2.3 Real-time tonal analysis

As this part is not yet ready it will be discussed just shortly. The analysis is divided into two parts, namely a pattern matching algorithm and a key-finding algorithm. An adequate implementation of the first depends on the latter, which will be explained in the following subsection. An important reason for separating both is one of not wasting CPU usage. For an analysis of real-time pitch spelling, as would be needed for automatic mapping of the [midi2ts] class for example, pattern matching would in many cases be superfluous. For a conversion of MIDI note numbers to a three-dimensional $Tts()$ value, however, additional data are necessary to avoid tuning problems concerning the syntonic comma as much as possible.

2.3.1 Pattern matching using the hexachord

The pattern chosen for the algorithm is the hexachord. Apart from just the fact that the hexachord is an important entity in medieval music theory it is especially valuable for tuning purposes. The hexachord was an attribute for medieval singers to intonate their pitches using reference notes at consonant distances (see Helmholtz, 1896, ch. 18). How this tuning could or should be performed will be explained in a future publication. To function properly the pattern matching algorithm itself needs any reference in some cases, especially when a choice between two distantly related hexachords has to be made. Such choices determine the experienced context and should therefore strongly depend on cognitive models. The algorithm needs reference and therefore memory to steer in the right direction. For example, a skip of a diminished seventh would always be interpreted as a major sixth, if a decision were not to be made by using memory and a 'feeling' of reference notes, from which one departed. Such reference could be offered by a separate algorithm, e.g. a key-finding algorithm.

2.3.2 A key-finding algorithm

Key-finding, just as pitch spelling algorithms already exist in several forms. For example some of them are discussed by David Temperley, offering his own implementation as well (Temperley, 2004, ch. 7). Another algorithm is proposed by Robert Rowe (Rowe, 2001, ch. 2). For the purpose of this research project, however, a result that just hints in the right direction would probably be sufficient. At least memory should be part of the algorithm, though that might be obvious.

3 Conclusions

Tuning classes - to be bundled in a software library - for at least the $Ts()$ based system are completely elaborated, except for a modulation automation function. However, such, based on a just a simple real-time key-finding algorithm, could be implemented soon after just a little bit more research and coding. The ideas for a more sophisticated three-dimensional tuning system, suited for five-limit just intonation, are at this moment reaching the point of first implementation and testing. A just intonation implementation, based on just key (mode and tonic) information, has been tested already, although this has not been mentioned yet. Graphical examples of how hexachords relate to each other and how they can be tuned, music examples that illustrate some of the requirements the algorithms have to meet, all this is or has been developed. The output of this research will be published in a future article that will mainly focus on the combination of tuning and cognition models. All implemenations will be part of the tuning library *Muditulib*, of which the initial version will be released soon.

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A Tritones

As has been said in section 2.1 the tritone and its octave inversion[13] are generally considered dissonant intervals, although they might be called con-

[13] I.e. the augmented fourth and the diminished fifth

sonants if intonated according to a 7-limit just intonation, that is $7:5$ and of course $10:7$ for its inversion. As Fokker describes the tuning of 31 equal dieses[14], rather than using mathematical equations, he shows that this pure tritone fits almost perfectly into this tuning system by a figure (see Fokker and Pol, 1942). Besides the question if this tritone then also approximately fits into the $Ts()$ based tuning system with r set to $\frac{3}{5}$ this raises the idea of calculating the perfect ratio r for the tritone and the octave. Counting the dieses in the figure this approximation of the pure tritone turns out to be equal to three whole tones, which is $3 \cdot 5 = 15$ dieses. This means that the tritone and its inversion are both part of the diatonic interpretation of 31-TET, so they are regularly used without the need of obscure modulations or typical alterations of *microtonalism*[15]. This is confirmed by the calculation of the *tritone temperament*[16], which is a two-dimensional tuning that tempers all intervals in favor of the tritone and octave. Equation 18 shows the relation to r. Equation 19 then shows that this temperament is pretty close to 31-TET ($r = 0.6$) and can be considered as belonging to the region of mean tone temperaments. As a matter of fact, if r is larger than $\frac{1}{2}$, the tritone is smaller than its pure $(2:1)$ octave inversion and vice-versa. If for example in 53-TET the amounts of commas for s and T are 4 and 9, respectively, the tritone is larger and consists of 27 commas, its inversion of 26. On the other hand, if the three-dimensional division of $5:8:9$ for $s:t:T$ is chosen, the tritone consists of 26 commas and its inversion of 27.

$$\frac{7}{5} = 2^{(3/(5+2 \cdot r))} \tag{18}$$

$$r \approx 0.590064075659218 \tag{19}$$

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[14]This means 31-TET. The original diesis is the leftover of one octave minus three major thirds ($2^7 \cdot 5^{-3} = \frac{128}{125}$), the interval *His* to *C* in mean tone temperament. As it is pretty close to mean tone temperament Fokker uses the same term for all equal parts in 31-TET

[15]Here, with the term microtonalism, an apparent urge of some composers to choose tones in extreme favor of harmony and at a huge expense of scale and melody is meant.

[16]No reference to a previous mention or calculation of this temperament is known to the present writer. He, however, dares not state it doesn't exist.

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